iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Wilson-Feynman graph technique and universality

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1976 J. Phys. A: Math. Gen. 9 1913

(http://iopscience.iop.org/0305-4470/9/11/016)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.88
The article was downloaded on 02/06/2010 at 05:13

Please note that terms and conditions apply.

J. Phys. A: Math. Gen., Vol. 9, No. 11, 1976. Printed in Great Britain. © 1976

Wilson–Feynman graph technique and universality

M Combescot† and M Droz‡

† Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853, USA and Groupe de Physique du Solide de l'Ecole Normale Supérieure, Tour 23, 2 Place Jussieu 75221, Paris Cedex 05, France

‡ Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853, USA and Physics Department, Temple University, Philadelphia, Pa 19122, USA

Received 16 March 1976, in final form 21 June 1976

Abstract. The critical exponents and scaling function are calculated with the Wilson–Feynman graph technique to second order in $\epsilon = 4-d$ for an n-component system with anisotropic Brillouin zone, using a general cut-off for the bare propagator $G(\boldsymbol{p}, r_0, \lambda) = (r_0+p^2)^{-1}f(\boldsymbol{p}/\lambda, r_0/\lambda^2)$. Universality holds provided that $f(\boldsymbol{v}, x)$ is analytic in $\boldsymbol{v}$ and x around $(\boldsymbol{0}, 0)$ and the first derivative $f'_{\boldsymbol{v}}(\boldsymbol{0}, x) = 0$.

1. Introduction

One of the most striking aspects of the critical behaviour of physical systems is the notion of 'universality'. Experimental results and theoretical investigations show (Fisher 1974) that the critical behaviour of a system can be described in terms of a few parameters (such as critical indices and scaling function) which depend only on the dimensionality d, the number of components of the order parameter n, the range of the interactions σ and the symmetry group of the system $\mathscr{G}$ (Mukamel and Krinsky 1976). The introduction of the renormalization group technique made it possible to give a coherent description of the concept of universality in terms of relevant, marginal or irrelevant scaling and redundant operators (Green 1976, Wegner 1974).

However, an exact renormalization group analysis becomes generally intractable in calculation beyond the lowest order in perturbation theory (Bruce *et al* 1974). Particularly, in the framework of the so called $\epsilon = 4-d$ expansion (d being the dimensionality of the system), irrelevant variables enter beyond the first order in ϵ in a complicated way. Accordingly, most calculations use the simpler Wilson–Feynman graph technique (WFGT) (Wilson and Kogut 1974, Wilson 1972).

In both methods it is assumed that only fluctuations with long wavelength (or small momentum) play a role in the critical domain. Thus the bare propagator $G_\theta^{-1}(q, r) = (r+q^2)\theta^{-1}(\lambda^2-q^2)$ essentially has been used, which corresponds to taking a spherical Brillouin zone. As has been previously stated, it is believed that the shape of the Brillouin zone as well as higher powers of q in the propagator are irrelevant. Thus the calculation with a bare propagator which contains an arbitrary cut-off function is not expected to change any of the universal quantities.

This is an important point in the renormalization group explanation of the universality. Many works have been devoted to this question within different frameworks. Recently, the universality of the critical exponent η has been studied and discussed within the framework of the general Gaussian renormalization group (Shukla and Green 1974, 1975, Rudnick 1975, Goldner and Riedel 1975). These renormalization group transformations simulate a class of symmetric smooth cut-offs. Even for this particular case, the disappearance of the cut-off is far from obvious. Abstract arguments (Schroer 1973, Mitter 1973) have been given in the framework of the renormalizability of the ϕ^4 theory to show the disappearance of the cut-off in the final results. However, it is not clear then that the results can be extended to the WFGT which mixes results coming from an exact renormalization group analysis with those of a perturbation scheme (Wilson and Kogut 1974). Moreover, the cut-offs used in the ϕ^4 field theory and in statistical physics have a different meaning. In field theory cut-offs are introduced in an *ad hoc* manner in order to regularize divergent integrals. Thus all the classes of cut-off studied have a spherical symmetry. For solid state physicists interested in the universal properties for a physical system at criticality, the cut-off has a physical meaning, since it describes the shape of the Brillouin zone which is never spherical. Consequently we find it useful to give a direct proof of the cut-off independence of the universal quantities within the framework of the widely used WFGT.

The purpose of this paper is to define a 'good' general cut-off, eventually anisotropic, for which the critical exponents and scaling function are indeed universal. The study is made for an n-component spin system up to order ϵ^2. We show that universality holds providing that the cut-off function $f(\boldsymbol{p}, x)$ (defined more precisely in § 2) is analytic in $\boldsymbol{p}$ and x around $(\boldsymbol{0}, 0)$ and its first derivative with respect to $\boldsymbol{p}$ is zero at the origin. The last condition ensures that there is no term linear in $\boldsymbol{p}$ in the inverse propagator, which is the expected situation for short-range forces.

The plan of the paper is as follows. In § 2, the model Hamiltonian with the most general form of cut-off in the bare propagator is defined; the principle of the Wilson–Feynman graph expansion is recalled. In §§ 3 and 4, the calculations of the critical exponent η and the four-point coupling constant u_{0c} are performed. It is shown that up to second order in ϵ, η is indeed cut-off independent. In § 5, the calculation of the critical index γ from the two-point correlation function with composite s^2 operator is given and again γ is found to be cut-off independent up to ϵ^2. Thus, hyperscaling implies that all critical exponents are indeed universal up to order ϵ^2. In § 6, the scaling function is shown to be universal up to ϵ^2. Details of the calculation of several graphs are given in appendixes 1–5.

2. The Wilson–Feynman graph technique

The basic idea of the WFGT is that, when d is close to 4, one can match the irreducible four-point interaction u_{R} with its perturbative expansion, providing that one chooses the bare four-point interaction as $u_{0c}(\epsilon)$, the bare inverse susceptibility as $r_0 = r_{0c}(\epsilon)$ and sets all the higher-order coupling constants equal to zero (Wilson and Kogut 1974).

For T close to T_c, the bare inverse susceptibility r_0 is such that $r - r_{0c} \sim T - T_c$.

Usually, the bare propagator $G_\theta(q, r_0) = (r_0 + q^2)^{-1}\theta(\lambda^2 - q^2)$ is used in the perturbation theory. The cut-off λ, proportional to the inverse lattice spacing, is such that $r_0 \ll \lambda^2$. The eventual anisotropy of the Brillouin zone, as well as higher-order terms in q in the propagator are presupposed to be irrelevant. In order to prove this point in general, we introduce a bare propagator with a general cut-off function f instead of a simple θ function. By definition, f has to be dimensionless. However, it could depend on the momentum $\boldsymbol{p}$, the inverse bare susceptibility r_0 and another momentum λ which characterizes the size of the cut-off (such that $f \sim 1$ for $p \ll \lambda$ and $f \sim 0$ for $p \gg \lambda$). So the general form for f is $f(\boldsymbol{p}/\lambda, r_0/\lambda^2)$. Moreover, it is physically reasonable to restrict ourselves to functions $f(\boldsymbol{v}, x)$ which are analytical around $(\mathbf{0}, 0)$ in order to avoid spurious non-analytical behaviour coming from the starting Hamiltonian itself. So the most general bare propagator is of the form

$$G_f^{-1}(\boldsymbol{p}, r) = (r + p^2) f^{-1}(\boldsymbol{p}/\lambda, r/\lambda^2) \tag{2.1}$$

and the Hamiltonian of the system is

$$H = \tfrac{1}{2} \sum_{\boldsymbol{p}} (r_0 + p^2) f^{-1}(\boldsymbol{p}/\lambda, r_0/\lambda^2) \boldsymbol{s}(\boldsymbol{p}) \cdot \boldsymbol{s}(-\boldsymbol{p}) + u_{0c} \sum_{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3} \boldsymbol{s}(\boldsymbol{p}_1) \cdot \boldsymbol{s}(\boldsymbol{p}_2) \boldsymbol{s}(\boldsymbol{p}_3) \cdot \boldsymbol{s}(-\boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3). \tag{2.2}$$

As usual, it is useful to introduce the real inverse susceptibility r, by performing a self-mass correction (which here is momentum dependent), namely:

$$\begin{aligned} H = {} & \tfrac{1}{2} \sum_{\boldsymbol{p}} (r + p^2) f^{-1}(\boldsymbol{p}/\lambda, r_0/\lambda^2) \boldsymbol{s}(\boldsymbol{p}) \cdot \boldsymbol{s}(-\boldsymbol{p}) \\ & + u_{0c} \sum_{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3} \boldsymbol{s}(\boldsymbol{p}_1) \cdot \boldsymbol{s}(\boldsymbol{p}_2) \boldsymbol{s}(\boldsymbol{p}_3) \cdot \boldsymbol{s}(-\boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3) \\ & + \tfrac{1}{2}(r_0 - r) \sum_{\boldsymbol{p}} f^{-1}(\boldsymbol{p}/\lambda, r_0/\lambda^2) \boldsymbol{s}(\boldsymbol{p}) \cdot \boldsymbol{s}(-\boldsymbol{p}). \end{aligned} \tag{2.3}$$

As $T \to T_c$, $r_0 \to r_{0c} \sim O(\epsilon)$ and $r \to 0$. So, close to T_c, $(r_0 - r) \sim O(\epsilon)$ and $u_{0c} \sim O(\epsilon)$, which allows one to use a perturbation theory in $(r_0 - r)$ and u_{0c} with

$$G^{-1}(\boldsymbol{q}, r, x_0) = (r + q^2) f^{-1}(\boldsymbol{v}, x_0) \tag{2.4}$$

as the bare propagator, where $\boldsymbol{v} = \boldsymbol{p}/\lambda$ and $x_0 = r_0/\lambda^2$.

Note that it suffices to introduce the cut-off in the quadratic part of the Hamiltonian (2.2) to get rid of all ultraviolet divergencies in the calculations, and thus, get the correct critical behaviour as $r \to 0$.

3. Universality of the critical exponent η

The critical exponent η is defined through the behaviour of the renormalized two-point function $\tilde{G}(\boldsymbol{q}, T)$ at criticality (Fisher and Jasnow 1976):

$$\tilde{G}^{-1}(\boldsymbol{q}, T_c) \sim q^{2-\eta}. \tag{3.1}$$

Using the Dyson equation which relates $\tilde{G}^{-1}$ to the bare propagator G^{-1} (see figure 1) and noting that $\Sigma(0, T_c) = 0$ ($\Sigma(\boldsymbol{q}, T)$ being the self-energy), we get

$$\begin{aligned} q^{2-\eta} \sim {} & q^2 f^{-1}(\boldsymbol{v}, x_{0c}) + r_{0c}(f^{-1}(\boldsymbol{v}, x_{0c}) - f^{-1}(0, x_{0c})) \\ & + 2^5(n+2) u_{0c}^2 (B(q, r = 0, x_{0c}) - B(0, 0, x_{0c})) + O(\epsilon^3) \end{aligned} \tag{3.2}$$

with

$$B(q, r, x_0) = \int \mathrm{d}\Omega_q \int \frac{\mathrm{d}^4 p \, \mathrm{d}^4 p'}{(2\pi)^8} G(\boldsymbol{p}, r, x_0) G(\boldsymbol{p}', r, x_0) G(\boldsymbol{p} + \boldsymbol{p}' + \boldsymbol{q}, r, x_0) \left(\int \mathrm{d}\Omega_q \right)^{-1} \tag{3.3}$$

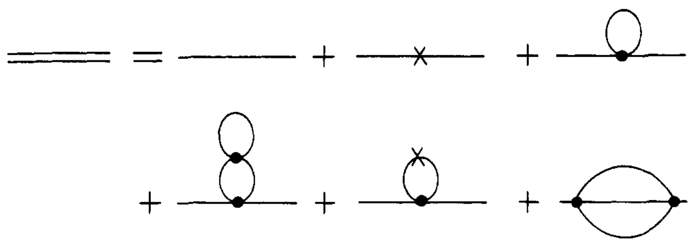

Figure 1. Diagrammatic expansion for the two-point correlation function ($\bullet = u_{0c}$, $\times = (r_0 - r)$).

where $\int d\Omega_q$ stands for the angular integration on $\boldsymbol{q}$. As we are working in the limit $q \ll \lambda$, we can neglect the v dependence in the left-hand side of (3.2). Moreover, the x_{0c} dependences of the B's in (3.2) give corrections of at least order ϵ^3. Dividing (3.2) by $f^{-1}(0, x_{0c}) = 1 + O(\epsilon)$ we get finally

$$q^{2-\eta} \sim q^2 - 2^5(n+2)u_{0c}^2(B(q,0,0) - B(0,0,0)) + O(\epsilon^3). \tag{3.4}$$

To go further, we have to calculate u_{0c} to order ϵ and $B(q,0,0) - B(0,0,0)$ to order ϵ^0. To calculate u_{0c} we match the irreducible four-point function u_R which behaves like $r^{(\epsilon-2\eta)/(2-\eta)}$ (Wilson 1972) with its diagrammatic expansion (see figure 2). We find

$$r^{(\epsilon-2\eta)/(2-\eta)} \sim u_{0c} - 4(n+8)u_{0c}^2 E_4(r, x_0) + O(u_{0c}^3, ru_{0c}^2) \tag{3.5}$$

with

$$E_d(r, x_0) = \int \frac{d^d p}{(2\pi)^d} G^2(\boldsymbol{p}, r, x_0) \tag{3.6}$$

η being of order ϵ^2 (see (3.5)). On calculating E_4 one finds (see appendix 2):

$$E_4(r, x_0) \sim -\tfrac{1}{2}K_4 \ln(r/\lambda^2) + O(r^0, \epsilon) \tag{3.7}$$

where $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$. One also finds from (3.5) that u_{0c} is indeed universal to order ϵ, namely,

$$u_{0c} = \frac{\epsilon}{4(n+8)K_4} + O(\epsilon^2). \tag{3.8}$$

On the other hand, it is shown in appendix 3 that

$$B(q,0,0) - B(0,0,0) = U(q) + q^2\mathcal{N} \tag{3.9}$$

where $U(q)$ is a universal function of q and $\mathcal{N}$ a non-universal constant. Thus the only non-universal part which is coming in (3.4) is of the form $\epsilon^2 q^2 \mathcal{N}$, so that $q^{2-\eta} \sim q^2(1 + \epsilon^2\mathcal{N}) + \epsilon^2$(universal function). Thus η is indeed universal.

4. Calculation of u_{0c} to order ϵ^2

In order to compute the susceptibility exponent γ and the two-point correlation function, we need to know u_{0c} up to order ϵ^2. Keeping graphs up to order ϵ^3 (see figure 2), (3.5) reads:

$$\begin{aligned} r^{\frac{1}{2}\epsilon-\eta} \sim & -u_{0c} + 4(n+8)u_{0c}^2 E_d(r, x_0) - 8(n+8)u_{0c}^2[4(n+2)u_{0c}A_4 C_3(r) + (r_0 - r)C_4(r)] \\ & - 16(n^2+6n+20)u_{0c}^3 E_4^2(r, 0) - 64(5n+22)u_{0c}^3 D(r) + O(r, \epsilon^4) \end{aligned} \tag{4.1}$$

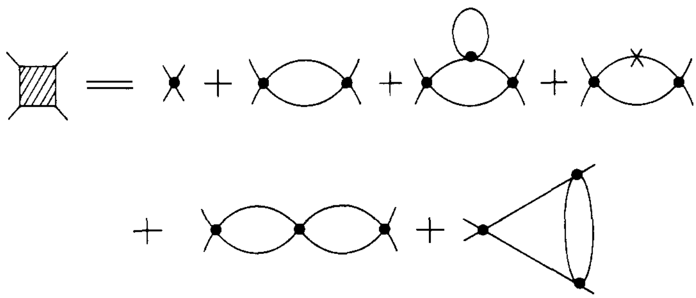

Figure 2. Diagrammatic expansion for the four-point irreducible interaction u_R.

where

$$A_4 = \int \frac{d^4p}{(2\pi)^4} G(\boldsymbol{p}, r, x_0) \tag{4.2}$$

$$C_i(r) = \int \frac{d^4p}{(2\pi)^4} \frac{f^i(\boldsymbol{v}, 0)}{(r+p^2)^3} \tag{4.3}$$

$$D(r) = \int \frac{d^4p\, d^4p'}{(2\pi)^8} G^2(\boldsymbol{p}, r, x_0) G(\boldsymbol{p}', r, x_0) G(\boldsymbol{p}+\boldsymbol{p}', r, x_0). \tag{4.4}$$

Moreover, (3.4) gives

$$r = r_0 f^{-1}(0, x_0) + 4(n+2) u_{0c} A_4 + O(\epsilon u_{0c}, r u_{0c}, u_{0c}^2). \tag{4.5}$$

The integrals A_4, $C_i(r)$, $D(r)$ are calculated in appendixes 1, 4 and 5 using the assumption that $f(\boldsymbol{v}, x)$ is analytic around $(\boldsymbol{0}, 0)$. The matching condition (4.1) can be fulfilled providing that $f'_v(\boldsymbol{v}, 0) = 0$ (see calculation of C_i). This condition implies that there is no linear term in $\boldsymbol{p}$ in the bare propagator as should be the case for short-range forces. Finally, the solution of (4.1) is:

$$u_{0c} = \frac{\epsilon}{4K_d(n+8)} \lambda^{\epsilon} (1 + \epsilon \bar{u}) \tag{4.6}$$

with

$$\bar{u} = \frac{3(3n+14)}{(n+8)^2} + \tilde{B} + \frac{2(n+2)}{n+8} \tilde{A} (f'_x(\boldsymbol{0}, 0) - \tfrac{1}{2}\lambda^2 f''_v(\boldsymbol{0}, 0)) \tag{4.7}$$

where

$$\tilde{B} = \int_0^{\infty} \frac{dv}{v} (\hat{f}^2(v, 0) - \theta(1-v)), \tag{4.8}$$

$\hat{f}(v, 0)$ being the angular average of $f(\boldsymbol{v}, 0)$ and $\tilde{A}$ a non-universal constant.

It is easy to verify that if $f(\boldsymbol{v}, x) = \theta(1-v)$, (4.6) and (4.7) lead to the usual result (Bruce *et al* 1974).

5. Universality of the susceptibility exponent γ

The direct but naive way to calculate γ is by computing the susceptibility directly, which leads to (Wilson 1972):

$$\Gamma^{(1/\gamma)-1} \sim 1 + r^{-1}(\Sigma(q=0, r) - \Sigma(q=0, r=0)). \tag{5.1}$$

If it is relatively easy to extract the leading divergent terms of $\Sigma(0, r) - \Sigma(0, 0)$, it is much more difficult to determine the contributions of order $O(r)$, needed to calculate γ up to order ϵ^2 from (5.1). In fact, the best way to calculate γ is to look at the two-point correlation function with composite s^2 operator (Nickel 1974). Let us define:

$$\Gamma_{2s}(r, \boldsymbol{q}_i) = \tfrac{1}{2}(\tilde{G}(r, \boldsymbol{q}_1)\tilde{G}(r, \boldsymbol{q}_2))^{-1} \int \mathrm{d}^d x_1 \, \mathrm{d}^d x_2$$

$$\times \left\langle s_1(\boldsymbol{x}_1) s_1(\boldsymbol{x}_2) \sum_{i=1}^{n} s_i^2(0) \right\rangle_{\mathrm{c}} \exp[\mathrm{i}(\boldsymbol{q}_1 \cdot \boldsymbol{x}_1 + \boldsymbol{q}_2 \cdot \boldsymbol{x}_2)] \tag{5.2}$$

$\langle \ldots \rangle_{\mathrm{c}}$ meaning average for the connected graphs only. Then, one can show (Brézin *et al* 1973) that:

$$\Gamma_{2s}(r, \boldsymbol{q}_i = 0) \overset{r \to 0}{\sim} r^{1-(1/\gamma)}. \tag{5.3}$$

The diagrammatic expansion of Γ_{2s} is very similar to the one for u_{R} (see figure 3), namely:

$$\begin{aligned}\Gamma_{2s}(r, \boldsymbol{q}_i = 0) \sim u_{0\mathrm{c}} &- 4(n+8)u_{0\mathrm{c}}^2 E_d(r, x_0) \\ &+ 8(n+8)u_{0\mathrm{c}}^2[4(n+2)u_{0\mathrm{c}}A_4 C_3(r) + (r_0 - r)C_4(r)] \\ &+ 16(n+2)^2 u_{0\mathrm{c}}^3 E_4^2(r, 0) + 96(n+2)u_{0\mathrm{c}}^3 D(r).\end{aligned} \tag{5.4}$$

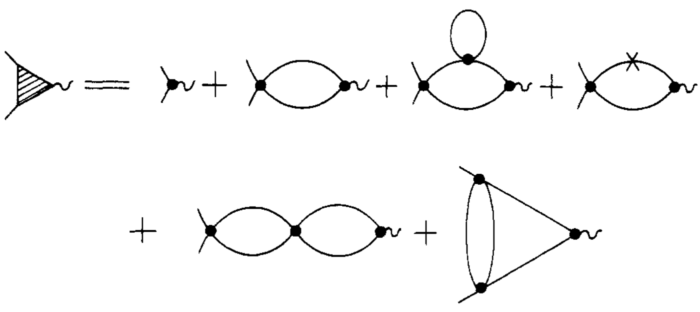

Figure 3. Diagrammatic expansion for the two-point correlation function with composite s^2 operator.

All the graphs appearing in (5.4) have already been calculated for u_{R}, which makes the computation of γ trivial now. Using the values of the various graphs calculated in appendixes 1, 2, 4, 5 and the value of $u_{0\mathrm{c}}$ given by (4.6) and (4.7), one can check that all the dependences on λ and $f(\boldsymbol{v}, x)$ cancel out in the expression for γ.

6. Universality of the scaling function

For $q \ll \lambda$ the correlation scaling hypothesis asserts that for $T > T_{\mathrm{c}}$, the two-point correlation function can be written asymptotically as

$$\tilde{G}(\boldsymbol{q}, T) \simeq rD(q\xi), \tag{6.1}$$

ξ being the second moment correlation length. $D(y = q\xi)$ is the scaling function. The scaling function has been studied in detail up to order ϵ^2 for some particular cut-offs (Combescot *et al* 1975, Fisher and Aharony 1973) and has been shown to be independent of the size of the cut-off in these particular cases. The purpose of this section is to show that $D(y)$ is also independent of the shape $f(\boldsymbol{v}, x)$ to order ϵ^2. Subtracting $\tilde{G}^{-1}(q = 0, r)$ from $\tilde{G}^{-1}(q, r)$ and using for Σ the diagrammatic expansion shown in figure 1 we get:

$$\tilde{G}^{-1}(\boldsymbol{q}, r) - r = (r + q^2)f^{-1}(\boldsymbol{v}, x_0) - rf^{-1}(0, x_0) - (r_0 - r)(f^{-1}(\boldsymbol{v}, x_0) - f^{-1}(0, x_0))$$
$$+ 2^5(n+2)u_{0c}^2(B(q, r, x_0) - B(0, r, x_0)) + \mathrm{O}(\epsilon^3), \qquad (6.2)$$

B being defined by (3.3). As we are interested in the asymptotic limit $v \to 0$, we can set $v = 0$ in (6.2). The last term of the right-hand side of (6.2) can be evaluated by a similar procedure to the one in appendix 3. One can show (Combescot *et al* 1975) that the only non-universal part coming from the difference of the B's is a non-universal constant-time q^2.

So the coefficient of q^2 in the right-hand side of (6.2) is $f^{-1}(0, x_0) + \epsilon^2$(non-universal) $= 1 + \mathrm{O}(\epsilon)$ and thus it will not affect the universal part of the result to order ϵ^2. Consequently, no cut-off-dependent part remains in the scaling function $D(y)$.

7. Conclusions

We have shown, in the framework of the WFGT (for an n-component Landau–Ginsburg model), the properties that the anisotropic cut-off function $f(\boldsymbol{v}, x)$, contained in the bare propagator, has to have in order to lead to universal results. We have shown that if the cut-off $f(\boldsymbol{v}, x)$ is analytic around $(\mathbf{0}, 0)$ and $f'_{\boldsymbol{v}}(\mathbf{0}, x) = 0$, the critical exponents η and γ are universal. Hyperscaling then implies that all the critical exponents are universal. Moreover, the scaling function around T_c is also shown to be universal. In a physical system, the cut-off function mimics the shape of the (anisotropic) Brillouin zone. The analyticity of $f(\boldsymbol{v}, x)$ in $\boldsymbol{v}$ expresses the fact that we are dealing with a 'good' d-dimensional Brillouin zone. The Brillouin zone can be extremely anisotropic, but cannot 'collapse' in one direction to a $(d-1)$-dimensional zone. The analyticity of $f(\boldsymbol{v}, x)$ in x expresses the fact that the bare Hamiltonian does not have some singular temperature dependence. Finally, the condition $f'_{\boldsymbol{v}}(\mathbf{0}, x) = 0$, expresses the fact that the bare propagator does not have a linear term in $\boldsymbol{p}$, which is demanded by the short-range character of the interaction. The above conditions on $f(\boldsymbol{v}, x)$ are thus physically very reasonable and are fulfilled for any physical system described by a $\Lambda\phi^4$-like Hamiltonian.

Note that the above verification has been shown up to order ϵ^2 only. However, the structure of the calculation suggests that the conclusions are valid to all order in ϵ.

Acknowledgments

The authors would like to thank Dr T Bell, Professors M E Fisher, M S Green and K G Wilson for many helpful comments. This work was partially supported by the NSF, grant No. GP41251X. MD wishes to acknowledge financial support from the Swiss National Foundation for Scientific Research.

Appendix 1

A_4, defined by (4.2), can be written as:

$$A_4=\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{f(\boldsymbol{v}, x_0)}{r+p^2}=K_4\lambda^2\int \frac{v^3\,\mathrm{d}v}{x+v^2}f(\boldsymbol{v}, x_0). \qquad (\mathrm{A}.1)$$

$f(0, 0)$ is equal to 1 so the function is convergent for $x=0$, and the denominator can be replaced by v^2 to order $\mathrm{O}(x)$. Since $f(\boldsymbol{v}, x)$ is analytical around $(\mathbf{0}, 0)$ the Euler–MacLaurin expansion

$$f(\boldsymbol{v}, x_0)=f(\boldsymbol{v}, 0)+x_0 f'_x(\boldsymbol{v}, hx_0) \qquad (\mathrm{A}.2)$$

shows that the second term f'_x gives a finite integral contributing to A_4 (assuming $f'_x(0, 0)$ exists) and so can be omitted to order x_0 (which is, from (4.5), of order r and u_{0c}). Finally, after performing the angular integration on the cut-off $f(\boldsymbol{v}, x)$ which defines

$$\hat{f}(v, x)=\frac{\int \mathrm{d}\Omega_v f(\boldsymbol{v}, x)}{\int \mathrm{d}\Omega_v} \qquad (\mathrm{A}.3)$$

one finds

$$A_4=K_4\lambda^2\int_0^\infty v\,\mathrm{d}v\hat{f}(v, 0)+\mathrm{O}(r, u_{0c}). \qquad (\mathrm{A}.4)$$

Appendix 2

Let us look at the calculation of $E_d(r, x_0)$ (defined by (3.6)), up to order ϵ. Using the analyticity of f and the fact that x_0 is of order ϵ, one can expand E_d as

$$E_d(r, x_0)=\int \frac{\mathrm{d}^dp}{(2\pi)^d}\frac{f^2(\boldsymbol{v}, 0)}{(r+p^2)^2}+2x_0\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{f(\boldsymbol{v}, 0)f'_x(\boldsymbol{v}, 0)}{(r+p^2)^2}+\mathrm{O}(\epsilon^2). \qquad (\mathrm{A}.5)$$

(i) Considering the first integral of the right-hand side, one can very quickly find its most divergent part when x goes to zero, saying that for $x=0$ the function behaves at the lower boundary like $(1-\epsilon\ln v)v^{-1}$, so one expects $\ln x+\epsilon\ln^2 x+\ldots$ to be the asymptotic behaviour of the integral. Unfortunately, one also needs the second most divergent term, so one has to do the calculation more carefully. One first performs the angular average and then writes $\hat{f}^2(v, 0)$ as $(\hat{f}^2(v, 0)-\theta(1-v))+\theta(1-v)$ in order to separate the terms coming from the slope of the cut-off, so that

$$\int \frac{\mathrm{d}^dp}{(2\pi)^d}\frac{f^2(\boldsymbol{v}, 0)}{(r+p^2)^2}=\lambda^{-\epsilon}K_d\left\{\int_0^1 \frac{v^3(1-\epsilon\ln p)}{(x+v^2)^2}\mathrm{d}v+\int_0^\infty \frac{v^3}{(x+v^2)^2}(\hat{f}^2(v, 0)-\theta(1-v))\right.$$
$$\left.-\epsilon\int_0^\infty \frac{v^3\ln v}{(x+v^2)^2}(\hat{f}^2(v, 0)-\theta(1-v)). \right. \qquad (\mathrm{A}.6)$$

Because $\hat{f}^2-\theta$ behaves at least as v for small $v(\hat{f}(0,0)=1$ and $\hat{f}(v,0)$ analytical), the last two integrals are convergent for $x=0$, so, finally, one gets

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{f^2(\boldsymbol{v},0)}{(r+p^2)^2} = \lambda^{-\epsilon} K_d[-\tfrac{1}{2}\ln x - \tfrac{1}{2} + \tilde{B} + \tfrac{1}{2}\epsilon(\tfrac{1}{4}\ln^2 x + \tfrac{1}{2}\ln x + \text{non-universal constant}) + \mathrm{O}(\epsilon^2)] \tag{A.7}$$

with

$$\tilde{B} = \int_0^\infty \frac{\mathrm{d}v}{v}(\hat{f}^2(v,0) - \theta(1-v)). \tag{A.8}$$

(ii) At the order of calculation necessary in this paper, we need only the divergent term, not the constant proportional to ϵ. Noting that x_0 is of order ϵ, $f(\mathbf{0},0)$ is equal to 1, one finds, in the second integral of (A.5), a term in $\ln x$ if $f'_x(\mathbf{0},0)\neq 0$. This term comes, as usual, from the lower boundary of the integral. Namely, we get

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{f(\boldsymbol{v},0) f'_x(\boldsymbol{v},0)}{(r+p^2)^2} = -\tfrac{1}{2} K_4 f'_x(0,0)\ln x + \text{constant}. \tag{A.9}$$

Finally, $E_d(r,x_0)$ reads

$$E_d(r,x_0) = K_d \lambda^{-\epsilon}[-\tfrac{1}{2}\ln x - \tfrac{1}{2} + \tilde{B} + \epsilon(\tfrac{1}{8}\ln^2 x + \tfrac{1}{4}\ln x) - x_0 f'_x(0,0)\ln x]. \tag{A.10}$$

Appendix 3

The easiest way to calculate $B(q, r=0, x_0=0) - B(0,0,0)$ (defined by (3.3)) is to replace the $G(\boldsymbol{p},0,0)$ by their Fourier transforms $G(\boldsymbol{x},0,0)$ so that

$$B(q,0,0) - B(0,0,0) = \frac{1}{\int \mathrm{d}\Omega_q} \int \mathrm{d}\Omega_q \int \mathrm{d}^4 x\, G^3(\boldsymbol{x},0,0)(\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} - 1) \tag{A.11}$$

and then to make the angular average on $\boldsymbol{q}$ which gives $-1+[2J_1(qx)/qx]$, the cut-off in x space operates now for small x. Since the Fourier transform of $G_0(p,r)=(r+p^2)^{-1}$ is

$$G_0(x,r) = \frac{\sqrt{r}}{(2\pi)^2} \frac{K_1(x\sqrt{r})}{x}, \tag{A.12}$$

the bare propagator $G(x,r,x_0)$ diverges like x^{-2} without the help of the cut-off. Thus, one can rewrite (A.11) in the form

$$B(q,0,0) - B(0,0,0) = \int \mathrm{d}^4 x\, G^3(\boldsymbol{x},0,0)\left(\frac{-q^2x^2}{8}\right) + \int \mathrm{d}^4 x G_0^3(x,0)\left(\frac{2J_1(qx)}{qx} - 1 + \frac{q^2x^2}{8}\right) + \mathrm{O}\left(\frac{q}{\lambda}\right). \tag{A.13}$$

The second integral does not depend on the cut-off, the first one gives a non-universal constant for the coefficient of q^2, so that $B(q,0,0) - B(0,0,0) = q^2$(non-universal constant) + universal function of $q \equiv q^2\mathcal{N} + U(q)$.

Appendix 4

It is convenient to calculate directly

$$\tilde{C}(x)=\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{f^3(\boldsymbol{v},0)}{(r+p^2)^3}[(r_0-r)f(\boldsymbol{v},0)+4(n+2)u_{0c}A_4] \tag{A.14}$$

because one knows from (4.5), that

$$r_0=-4(n+2)u_{0c}A_4+\mathrm{O}(\mathrm{r},\epsilon^2) \tag{A.15}$$

and $\tilde{C}(x)$ can be rewritten as

$$\tilde{C}(x)=r_0\int \frac{\mathrm{d}^4 v}{(2\pi)^4}\frac{f^3(\boldsymbol{v},0)}{(x+v^2)^2}(f(\boldsymbol{v},0)-1)+\mathrm{O}(r). \tag{A.16}$$

Looking at the behaviour of the integrand at the lower boundary and expanding $f(\boldsymbol{v},0)$ near $\mathbf{0}$ one sees that the most divergent part of $\tilde{C}(x)$ will be in $1/x$ if $f'_{\boldsymbol{v}}(\boldsymbol{v},0)\neq 0$ but in $\ln x$ if $f'_{\boldsymbol{v}}(\boldsymbol{v},0)=0$ and $f''_{\boldsymbol{v}}(\boldsymbol{v},0)\neq 0$. With a behaviour in $1/x$, it will be impossible to scale the renormalized four-point interaction u_{R} as usual, and the critical exponent would not be universal. So we impose $f'_{\boldsymbol{v}}(\boldsymbol{v},0)=0$ and $\tilde{C}(x)$ is given by

$$\begin{aligned}&\tilde{C}(x)=-\tfrac{1}{4}r_0K_4f''_{\boldsymbol{v}}(\mathbf{0},0)\ln x+\mathrm{O}(x^0,r,\epsilon^2)\\ &f'_{\boldsymbol{v}}(\mathbf{0},0)=0.\end{aligned} \tag{A.17}$$

Appendix 5

It is convenient to rewrite $D(r)$, defined by (4.4), as:

$$\begin{aligned}D(r)=&\int\frac{\mathrm{d}^4p\,\mathrm{d}^4p'}{(2\pi)^8}G^2(\boldsymbol{p},r,x_0)G(\boldsymbol{p}',r,x_0)(G(\boldsymbol{p}+\boldsymbol{p}',r,x_0)-G(\boldsymbol{p}',r,x_0))\\ &+\left(\int\frac{\mathrm{d}^4p}{(2\pi)^4}G^2(\boldsymbol{p},r,x_0)\right)^2.\end{aligned} \tag{A.18}$$

The second term is simply $B_4^2(r)$ and has already been calculated in appendix 2. The integration over $\boldsymbol{p}'$ of the first integral converges without the help of any cut-off so one can replace $G(\boldsymbol{p}',r,x_0)$ by $G_0(p',r)=(r+p^2)^{-1}$ and one finds that

$$\begin{aligned}&\int_0^\infty G_0(p',r)(G_0(p+p',r)-G_0(p',r))\frac{\mathrm{d}^4p'}{(2\pi)^4}\\ &\qquad=K_4\left[1-\frac{\surd(p^2+4r)}{p}\ln\left(\frac{p+\surd(p^2+4r)}{2\surd r}\right)\right].\end{aligned} \tag{A.19}$$

Inserting this value, the first integral is expected to behave like

$$\int v^{-1}(1-\ln v)\,\mathrm{d}v\sim\ln^2 x+\ldots.$$

A more precise calculation is done again writing $f^2 = f^2 - \theta + \theta$ so that

$$D(r) - B_4^2(r) = K_4^2 \left\{ \int_0^1 \frac{v^3 \mathrm{d}v}{(x+v^2)^2} \left[1 - \frac{\surd(4x+v^2)}{v} \ln\left(\frac{v + \surd(4x+v^2)}{2\surd x} \right) \right] \right\}$$

$$+ \int_0^\infty \frac{v^3 \, \mathrm{d}v}{(x+v^2)^2} \left[1 - \frac{\surd(4x+v^2)}{v} \ln\left(\frac{v+\surd(4x+v^2)}{2} \right) \right] (\hat{f}^2(v,0) - \theta(1-v))$$

$$+ \ln(\surd x) \int_0^\infty \frac{v^3 \, \mathrm{d}v}{(x+v^2)^2} \frac{\surd(4x+v^2)}{v} (\hat{f}^2(v,0) - \theta(1-v)). \tag{A.20}$$

The last two integrals are convergent for $x = 0$, because for small v, $\hat{f}^2(v, 0) - \theta(1-v)$ behaves like $v^2(f'_v(0, 0)$ being zero) and so will give a constant term in x^0 (the last term giving $\tilde{B}$ already found in (A.9)). The first integral does not depend on the shape of the cut-off and gives $[-\frac{1}{8}\ln^2 x - \frac{1}{2}\ln x + \mathrm{O}(x^0)]$, so that finally

$$D(r) = K_4^2[\tfrac{1}{8}\ln^2 x - \tfrac{1}{2}\tilde{B} \ln x + \mathrm{O}(x^0)]. \tag{A.21}$$

References

Brézin E, Le Guillou J C and Zinn-Justin J 1973 *Phys. Rev.* D **8** 434–46
Bruce A D, Droz M and Aharony A 1974 *J. Phys. C: Solid St. Phys.* **7** 3673–86
Combescot M, Droz M and Kosterlitz J M 1975 *Phys. Rev.* B **11** 4661–73
Fisher M E 1974 *Rev. Mod. Phys.* **46** 597–616
Fisher M E and Aharony A 1973 *Phys. Rev. Lett.* **31** 1238–42
Fisher M E and Jasnow D 1976 *Theory of Correlations in the Critical Region*, eds C Domb and M S Green (New York: Academic Press) to be published
Goldner G and Riedel E K 1975 *Phys. Rev. Lett.* **34** 171–3
Green M S 1976 to be published
Mitter P K 1973 *Phys. Rev.* B **7** 2927–42
Mukamel D and Krinsky S 1976 *Phys. Rev.* B **13** 5065–77
Nickel B 1974 *J. Phys. A: Math. Gen.* submitted for publication
Rudnick J 1975 *Phys. Rev. Lett.* **34** 438–440
Schroer B 1973 *Phys. Rev.* B **8** 4200–8
Shukla P and Green M S 1974 *Phys. Rev. Lett.* **33** 1263–5
—— 1975 *Phys. Rev. Lett.* **34** 436–8
Wegner F J 1974 *J. Phys. C: Solid St. Phys.* **7** 2098–108
Wilson K G 1972 *Phys. Rev. Lett.* **28** 548–51
Wilson K G and Kogut J 1974 *Phys. Rep.* **12** 75–199